3.7: Implicit Differentiation

We use implicit differentiation when it is implied that some variable is a function of the original variable.

Ex(1): Power Chain Rule.

$$\frac{d}{dx}(u^n) = nu^{n-1}\cdot\frac{du}{dx} = n\cdot u^{n-1}\cdot u'.$$ It is implied that u is a fcn of x.

In general, we will try to differentiate things like

$$x^3+y^3-9xy=0, \quad y^2-x=0, \quad x^2+y^2-25=0$$

Ex(2): Find $\frac{dy}{dx}$ when $y^2=x$.

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$$y^2 = x$$

take derivatives of both sides

Chain rule on LHS

$$2y\cdot y' = 1$$

$$y' = \frac{1}{2y}.$$ The only solutions are given

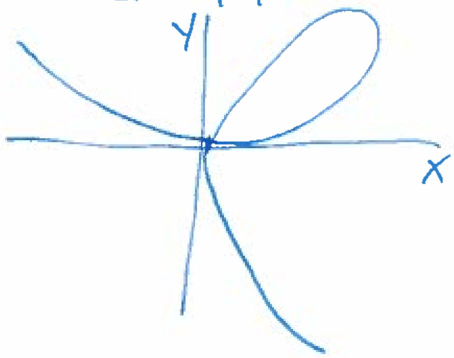
Clearly not a fcn of x

$$y_1 = \sqrt{x} \text{ or } y_2 = -\sqrt{x}.$$

we have $\frac{dy_1}{dx} = \frac{1}{2\sqrt{x}}$; $\frac{dy_2}{dx} = \frac{-1}{2\sqrt{x}}$.

Ex(3): $x^3+y^3-9xy=0$

Clearly y is not a fcn of x.



Take derivatives of both sides wrt x.

$$3x^2+3y^2\cdot y' - 9(y+x\cdot y') = 0$$

power chain rule   product rule

Ex(4): Find the slope of $x^2+y^2=25$ at $(3,-4)$

$$2x+2y\cdot y'=0$$

Evaluate at $(3,-4) \Rightarrow 2(3)+2(-4)\cdot y'=0$

$$y'=\frac{-6}{-8}=\frac{3}{4}.$$

Method for Implicit Differentiation

(1) Differentiate both sides of the eqn with respect to x, treating y as a differentiable fcn of x.

(2) Collect the terms and solve for $\frac{dy}{dx}$.

(Remark: Solving for $\frac{dy}{dx}$ is not always immediately obvious until Math 242)

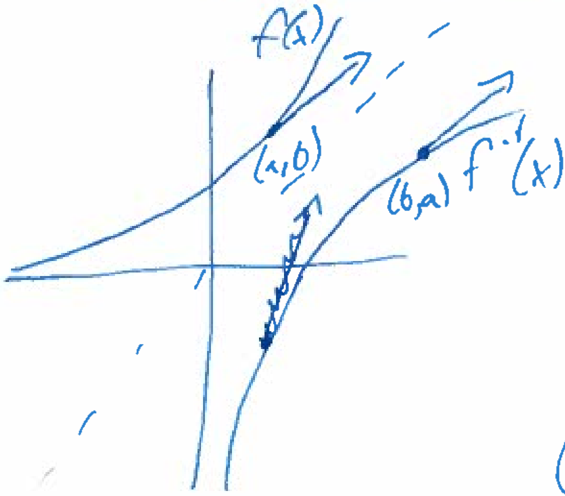
Ex(5): Find $\frac{d^2y}{dx^2}$ if $2x^3-3y^2=8$

1st derivative: $6x^2-6y\cdot y'=0$

2nd derivative: $12x-6(y'+y\cdot y'')=0$

$$y''=\frac{d^2y}{dx^2}=\frac{6y'-12x}{y}.$$

3.8: Derivatives of Inverse Fcns



How are the tangent lines?
~~match for a fixed value of x?~~

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{h\to 0} \frac{f(a+h)-b}{h}$$

$$(f^{-1})'(b) = \lim_{h\to 0} \frac{f^{-1}(b+h)-f^{-1}(b)}{h} = \lim_{h\to 0} \frac{f^{-1}(b+h)-a}{h}.$$

Ex(1): Let $f(x) = \frac{1}{2}x+1$. You can easily solve to get $f^{-1}(x) = 2x-2$.

So $f'(x) = \frac{1}{2}$ and $(f^{-1})'(x) = 2$. They are reciprocals!! :)

Theorem 3: The above holds in general when $f(a)=b$.

We have $(f^{-1})'(b) = \dfrac{1}{f'(a)}$.

or $(f^{-1})'(b) = \dfrac{1}{f'(f^{-1}(b))}$ by chain rule and the fact that $f^{-1}(f(a)) = a$.

Ex(2): Let $f(x) = x^2$, $x \geq 0$. $f^{-1}(x) = \sqrt{x}$. and $f'(x) = 2x$.

Clearly $(f^{-1})'(x) = \dfrac{1}{2\sqrt{x}} = \dfrac{1}{2\cdot f^{-1}(x)}$.

Natural Log: We already have seen $\frac{d}{dx}(\ln x) = \frac{1}{x}$.

Let's check this. $y = \ln x$ then its inverse is $\cancel{y}\,u = e^x$. and $\frac{du}{dx} = e^x$.

So $$\frac{dy}{dx} = \frac{1}{\frac{du}{dx}(\ln x)} = \frac{1}{e^{\ln x}} = \frac{1}{x}$$ as seen before.

Ex(3): Let k be a constant. Find $\frac{d}{dx}(\ln(kx))$.

Using the chain rule $\ln(kx) = \ln(u)$ where $u = kx$.

So $$\frac{d}{dx}(\ln(kx)) = \frac{1}{kx} \cdot k = \frac{1}{x}.$$

Ex(4): Use the chain rule to find $\frac{d}{dx}(e^{kx})$

$e^{kx} = e^u$ where $u = kx$.

So $\frac{d}{dx}(e^{kx}) = e^{kx} \cdot k = ke^{kx}$. (You can also see this with annoying use of power rule, power chain rule or inverse fcn rule.)

Ex(5): Let $f(x) = x^x$. Find $f'(x)$.

$f(x) = x^x = e^{x \cdot \ln x}$. ~~So f'(x)~~ Also $e^{x \cdot \ln x} = e^u$ where $u = x \cdot \ln x$.

So $$f'(x) = e^{x \cdot \ln x} \cdot \frac{du}{dx} = e^{x \cdot \ln x} \cdot (\ln x + x/x) = x^x(\ln x + 1).$$

Ex(6): Let a be a constant. Find $\frac{d}{dx}(a^x)$.

$$\frac{d}{dx}(a^x) = \frac{d}{dx}(e^{\ln a \cdot x}) = e^{\ln a \cdot x} \cdot \frac{d}{dx}(\ln a \cdot x) = \ln a \cdot a^x.$$